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# Two derivations of the master equation of quantum Brownian motion

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## Abstract

Central to many discussion of decoherence is a master equation for the reduced density matrix of a massive particle experiencing scattering from its surrounding environment, such as that of Joos and Zeh. Such master equations enjoy a close relationship with spontaneous localization models, like the GRW model. The aim of this paper is to present two derivations of the master equation. The first derivation is a pedagogical model designed to illustrate the origins of the master equation as simply as possible, focusing on physical principles and without the complications of  $S$ -matrix theory. This derivation may serve as a useful tutorial example for students attempting to learn this subject area. The second is the opposite: a very general derivation using non-relativistic many-body field theory. It reduces to the equation of the type given by Joos and Zeh in the one-particle sector, but correcting certain numerical factors which have recently become significant in connection with experimental tests of decoherence. This master equation also emphasizes the role of local number density as the ‘preferred basis’ for decoherence in this model.

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## 1. Introduction

Non-unitary master equations for a density matrix arise in both continuous state localization models, such as GRW theory [1], and in decoherence calculations in standard quantum theory. They differ, however, in their underlying physical pictures. GRW theory involves an explicitly modified dynamics in the Schrödinger equation. Standard decoherence calculations, by contrast, employ the usual Schrödinger dynamics for a system coupled to its environment, but this becomes a non-unitary dynamics for the reduced density matrix once the environment is traced out. These similarities and differences were discussed shortly after the appearance of GRW theory twenty years ago [2, 3], but there is surely still more to learn in this area.

It is a great pleasure to have the opportunity to contribute to this volume in honour of the 70th birthday of GianCarlo Ghirardi.

This paper concerns master equations in standard quantum mechanics of the type used in decoherence studies, obtained by coupling a point particle by some kind of environment. A simple example of such a master equation in one dimension is

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2M} \left( \frac{\partial^2 \rho}{\partial x^2} - \frac{\partial^2 \rho}{\partial y^2} \right) - D(x - y)^2 \rho. \quad (1.1)$$

This describes quantum Brownian motion for a free particle of mass  $M$  in the limit of negligible dissipation. This is the one-dimensional version of an equation first obtained by Joos and Zeh [4] (in the limit of small  $|x - y|$ ), which involves a massive free particle undergoing scatterings by an environment of much lighter particles. Many other derivations of this and similar equations have since been given [5–10]. This equation can also arise for the case of a point particle linearly coupled to a thermal bath of harmonic oscillators, and this model also has been the subject of many papers [11–13]. Models involving fields have also been considered [14–16]. (The literature on quantum Brownian motion is considerable so only a selection is mentioned here. See also [17].)

A key reference point in these studies is the Lindblad form of the master equation [18], which is the most general possible form a master equation can take under the assumption that the evolution is Markovian (a condition well satisfied in a wide variety of interesting models). The Lindblad master equation is

$$\frac{d\rho}{dt} = -i[H, \rho] - \frac{1}{2} \sum_{j=1}^n (\{L_j^\dagger L_j, \rho\} - 2L_j \rho L_j^\dagger). \quad (1.2)$$

Here,  $H$  is the Hamiltonian of the distinguished subsystem (sometimes modified by terms depending on the  $L_j$ ) and the  $n$  operators  $L_j$  model the effects of the environment. For example, the master equation of one-dimensional quantum Brownian motion, including dissipation, is of the Lindblad form with a single Lindblad operator

$$L = \left( \frac{4M\gamma kT}{\hbar^2} \right)^{\frac{1}{2}} x + i \left( \frac{\gamma}{2MkT} \right)^{\frac{1}{2}} p \quad (1.3)$$

as described in [19, 20]. (This reproduces equation (1.1) for small  $\gamma$  with  $D = 2M\gamma kT/\hbar^2$ ).

Equation (1.1) describes the decoherence process in which an arbitrary initial density matrix becomes approximately diagonal in position on a very short timescale. This process is thought to be a key element in understanding how classical behaviour emerges from quantum theory [21–23]. Recent experiments have been able to actually observe the rate of the decoherence process [24], which is connected to the constant  $D$  in equation (1.1). It turns out that some of the original derivations of the master equation led to incorrect values of  $D$ . More recent derivations [8, 14] have corrected these errors and produced the values of  $D$  compatible with experiments. (See also [25, 26].)

The purpose of this paper is to present two derivations of a class of master equations of the form equation (1.1) and its generalizations.

The first derivation, described in sections 2 and 3, is a simple pedagogical model, designed to illustrate the way in which the general form of equation (1.1) follows from some simple physical ideas. We therefore avoid the technical complications and sometimes non-transparent mathematical assumptions involved in these models, but make no claims about making physically accurate predictions. (This model is similar to that presented by Joos *et al* [27].)

The second derivation, described in sections 4, 5 and 6, is a very general derivation of a class of master equations and makes use of non-relativistic many-body field theory. It reduces, in form, to an equation of the type given by Joos and Zeh and others, but with a correct value of the decoherence rate that is compatible with experiments. This derivation is also relevant to another issue in decoherence theory, which is the question of the ‘preferred basis’—the natural basis in which interferences are destroyed. In the master equation (1.1), it is clearly the interferences between different values of *position* that are initially suppressed, so that position is the preferred basis. More generally, it is known that under evolution according to the Lindblad form equation (1.2), it is the Lindblad operators  $L_j$  that defined the preferred basis (at least in simple models) [28]. We will see in the many-body field theory model that most generally in system-environment models, *local number density* is the preferred basis (with position emerging as a special case of this in the one-particle sector). The special role of number density also echoes certain aspects of the GRW model [1]. Furthermore, local densities are thought to play a key role in the most general possible derivations of emergent classicality, even when there is no environment present [23, 29]

The many-body field theory derivation of the master equation first appeared, in essence, in [14], as a part of a wider investigation into the properties of the decoherent histories approach to quantum theory [30–34]. Here, it is presented on its own merits as a contribution to the theory of quantum Brownian motion.

## 2. A simple model

To set the context, we first briefly recall the basic ideas involved in decoherence. It is generally held that the emergence of classical behaviour for simple particle systems may take place when the system density matrix evolves from an initial pure state to a mixed state which is approximately diagonal in both position and momentum. This, loosely speaking, is what is meant by decoherence. Unitary evolution alone cannot produce such a transition, so it is necessary to consider a larger ‘universe’ consisting of the original system coupled to its surrounding environment. If the interactions between system and environment generate entanglement, then the reduced density matrix of the system only can evolve non-unitarily and produce the desired decoherence.

Turning to our specific model, we consider a system consisting of a particle in one dimension which interacts through occasional collisions with an environment consisting of gas of light particles. Except for the collisions, which are assumed to be very brief, the particles evolve freely. By considering the change in the system density matrix during these collisions we will derive the form of the decoherence term (the last term) in equation (1.1).

We first consider the collision process classically. Suppose the system particle has momentum  $P$  and mass  $M$  and a particle from the environment has momentum  $p$  and mass  $m$ . We will assume that the collision conserves both energy and momentum. If the final momenta are  $P'$  and  $p'$ , we therefore have

$$\frac{P^2}{2M} + \frac{p^2}{2m} = \frac{P'^2}{2M} + \frac{p'^2}{2m} \quad (2.1)$$

$$P + p = P' + p'. \quad (2.2)$$

Ignoring the trivial solution  $P = P'$  and  $p = p'$ , the final momenta are given by

$$P' = \frac{(M - m)}{(M + m)}P + \frac{2M}{(M + m)}p \quad (2.3)$$

$$\equiv aP + bp \quad (2.4)$$

$$p' = \frac{2m}{(M+m)}P - \frac{(M-m)}{(M+m)}p \quad (2.5)$$

$$\equiv cP - ap, \quad (2.6)$$

(where the coefficients  $a, b, c$  are read off from equations (2.3), (2.5)). It is now very useful to make two approximations. We first assume that the environment particles are much lighter than the system particles:

$$m \ll M. \quad (2.7)$$

Second, we assume that the speed of the system particle is much smaller than the speed of environment particles,

$$\frac{P}{M} \ll \frac{p}{m}, \quad (2.8)$$

(although the momenta  $P$  and  $p$  may be comparable in size). These approximations imply that the momenta after the collision are given by the much simpler expressions

$$P' \approx P + 2p \quad (2.9)$$

$$p' \approx -p \quad (2.10)$$

Turn now to the quantum case. We will make the reasonable assumption that energy and momentum are also conserved by the quantum description of the collision. The key idea in the quantum case is to work with states of definite momentum (plane wave states) and use the above results to deduce how they change during a collision. We will eventually also need to assume something about the locality of the interaction, but that will not be needed just yet.

Introducing the positions  $(x, q)$  of the system and environment particles, it follows from equations (2.9), (2.10) that an initial plane wave for the total system changes according to

$$e^{\frac{i}{\hbar}Px} e^{\frac{i}{\hbar}pq} \rightarrow e^{\frac{i}{\hbar}(P+2p)x} e^{-\frac{i}{\hbar}pq} \quad (2.11)$$

$$= e^{\frac{i}{\hbar}Px} e^{\frac{i}{\hbar}p(2x-q)} \quad (2.12)$$

(in the approximations described above). Or, in configuration space, the effect of the collision is to make the replacement

$$x \rightarrow x \quad (2.13)$$

$$q \rightarrow 2x - q \quad (2.14)$$

Since the plane waves are a complete set of states, this result determines the effects of a collision on any initial state.

We suppose that the initial density matrix of the whole system is a simple product state,  $\rho(x, y)\rho_{\mathcal{E}}(q, q')$ . As a result of a single collision, equations (2.13), (2.14) imply that the total density matrix changes according to

$$\rho(x, y)\rho_{\mathcal{E}}(q, q') \rightarrow \rho(x, y)\rho_{\mathcal{E}}(2x - q, 2y - q'). \quad (2.15)$$

Hence, the collision causes the system and environment to become entangled, as required for decoherence.

Tracing over the environment, we find that the density operator of the system only evolves according to

$$\rho(x, y) \rightarrow \rho(x, y) \int dq \rho_{\mathcal{E}}(2x - q, 2y - q). \quad (2.16)$$

We suppose that the collisions take place at a rate  $\Gamma$  per unit time. The total change in the density matrix of the system during a small time interval  $\Delta t$  is therefore given by

$$\Delta\rho(x, y) = -\Delta t F(x, y)\rho(x, y), \quad (2.17)$$

where

$$F(x, y) = \Gamma \left( 1 - \int dq \rho_{\mathcal{E}}(2x - q, 2y - q) \right). \quad (2.18)$$

This is also usefully written as

$$F(x, y) = \Gamma \text{Tr} \left( (1 - e^{2\frac{i}{\hbar}(x-y)\hat{p}}) \hat{\rho}_{\mathcal{E}} \right), \quad (2.19)$$

where  $\hat{p}$  is the momentum operator on the environment and the trace is over a complete set of environment states. It follows that the master equation is

$$\frac{\partial\rho}{\partial t} = \frac{i\hbar}{2M} \left( \frac{\partial^2\rho}{\partial x^2} - \frac{\partial^2\rho}{\partial y^2} \right) - F(x, y)\rho. \quad (2.20)$$

In the small  $|x - y|$  limit we have

$$F(x, y) = \frac{2\Gamma\langle\hat{p}^2\rangle}{\hbar^2}(x - y)^2, \quad (2.21)$$

where we have assumed that the environment state is such that  $\langle\hat{p}\rangle = 0$ . For a thermal environment state we have

$$\langle\hat{p}^2\rangle = mkT, \quad (2.22)$$

so we obtain a master equation of the form equation (1.1) with  $D = 2m\Gamma kT/\hbar^2$ .

We have therefore obtained the expected form of the master equation, using energy and momentum conservation, together with the approximations that the environment particles are much lighter and faster than the system particle.

A slightly different but simpler model along these lines was given by Joos *et al* [27]. Their model postulates a simple dynamics involving a particle being subject to random kicks, but without relating it to a collision process with energy and momentum conservation as here.

### 3. A Wigner function derivation including dissipation

It is now useful to give a more detailed derivation of the master equation in this simple model using the Wigner representation. We go beyond the approximations used above and work to leading order in  $m/M$ . This derivation shows how the dissipative terms arise and also establishes the connection between the collision rate  $\Gamma$  and the dissipation  $\gamma$ , hence connects microscopic and macroscopic parameters.

The Wigner function of the density matrix  $\rho(x, y)$  of a one-dimensional systems is defined by

$$W(p, q) = \frac{1}{2\pi\hbar} \int d\xi e^{-\frac{i}{\hbar}p\xi} \rho \left( q + \frac{1}{2}\xi, q - \frac{1}{2}\xi \right) \quad (3.1)$$

together with its inverse

$$\rho(x, y) = \int dp e^{\frac{i}{\hbar}p(x-y)} W \left( p, \frac{x+y}{2} \right). \quad (3.2)$$

The Wigner function has the properties

$$\int dp W(p, q) = \rho(q, q) \quad (3.3)$$

$$\int dq W(p, q) = \tilde{\rho}(p, p), \quad (3.4)$$

where  $\tilde{\rho}$  is the Fourier transform of  $\rho(x, y)$ , so  $W$  contains the usual position and momentum probabilities as its marginal distributions. The Wigner function is the closest thing quantum mechanics has to a phase space probability distribution function but narrowly fails since  $W$  is not always positive. Its time evolution is identical to classical evolution for linear systems, with corrections proportional to powers of  $\hbar^2$  for nonlinear potentials. It is therefore a very useful tool for discussing the connection between quantum and classical systems, although note that there are subtle differences between the Wigner function and a classical distribution function, as we shall see shortly. (See [35] for properties of the Wigner function.)

In the absence of interactions, the two-particle Wigner function  $W_2(P, X, p, q)$  obeys the equation

$$\frac{\partial W_2}{\partial t} = -\frac{P}{M} \frac{\partial W_2}{\partial X} - \frac{p}{m} \frac{\partial W_2}{\partial q}, \quad (3.5)$$

and this corresponds to unitary evolution of the density matrix. We need to find terms representing the collision to add to the right-hand side, using an argument along the lines of that used in the previous section for the density matrix. We again consider the collision process described by equations (2.4), (2.6) and look for the change in the Wigner function.

If  $W$  were a classical distribution function, it would be sufficient to consider only the momentum transfer described by equations (2.4), (2.6). In the Wigner function, however, things are a bit more subtle. If equations (2.4), (2.6) describe a process in the momentum representation of a quantum system, then there is a corresponding transformation in position space. Proceeding along lines identical to the derivation of equations (2.13), (2.14), it is easy to see that this transformation is

$$X \rightarrow aX + cq = X + c(q - X) \quad (3.6)$$

$$q \rightarrow bX - aq = 2X - q + c(q - X). \quad (3.7)$$

That is, the collision process in the Wigner function involves *both* the transformation equations (2.4), (2.6) on the momenta together with the transformation equations (3.6), (3.7) on the positions. This must be the case because on integrating out the momenta in the Wigner function, the correct distributions for position must be obtained and vice versa (as in equations (3.3), (3.4)).

Denoting the system Wigner function by  $W(P, X)$  and the environment Wigner function by  $W_{\mathcal{E}}(p, q)$ , the above discussion implies that the effect of a collision is to produce the transition

$$W(P, X)W_{\mathcal{E}}(p, q) \rightarrow W(aP + bp, aX + cq)W_{\mathcal{E}}(cP - ap, bX - aq) \quad (3.8)$$

in the two-particle Wigner function. However, in this derivation it turns out that it is also important to incorporate the fact that the interaction is local, that is, it is described by a potential of the form  $V(x - q)$  which decays for large  $|x - q|$ . In a more complete derivation, this would be accomplished by multiplying equation (3.8) by a function of  $X - q$  which is concentrated around  $X = q$ . However, in the interests of keeping the derivation heuristic,

we will incorporate this ‘by hand’, by assuming that terms of the form  $|q - X|$  are small in some sense. Since the constant  $c$  is also small (of order  $m/M$ ), the most minimal and natural implementation of locality is to simply drop the terms  $c(q - X)$  in equations (3.6), (3.7).

Integrating out the environment, and considering  $\Gamma$  collisions per unit time, we now have that the evolution equation of the system Wigner function is

$$\frac{\partial W}{\partial t} = -\frac{P}{M} \frac{\partial W}{\partial X} + \Gamma \int dp dq [W(aP + bp, X)W_\varepsilon(cP - ap, 2X - q) - W(P, X)W_\varepsilon(p, q)], \quad (3.9)$$

which is a Boltzmann equation. Using some simple changes of variable for  $p$  and  $q$  in the first term, this is easily rewritten,

$$\frac{\partial W}{\partial t} = -\frac{P}{M} \frac{\partial W}{\partial X} + \Gamma \int dp dq \left[ \frac{1}{a} W \left( \frac{P}{a} - \frac{b}{a} p, X \right) - W(P, X) \right] W_\varepsilon(p, q). \quad (3.10)$$

(From this form we see that the position coordinates in the Wigner function are, in the end, effectively unchanged as a result of the collision, despite the argument above; but this is due to the approximation of dropping terms proportional to  $c(q - X)$ ).

We again assume that the environment is in a thermal state at temperature  $T$ . To obtain the more familiar form of the equation, we will assume that  $W$  is slowly varying in  $P$  so that it may be expanded in derivatives. We will also take  $m \ll M$  and look for the leading-order terms in  $m/M$ , so we have that

$$\frac{1}{a} = \frac{M + m}{M - m} \approx 1 + \frac{2m}{M}. \quad (3.11)$$

We thus obtain

$$\frac{\partial W}{\partial t} = -\frac{P}{M} \frac{\partial W}{\partial X} + \Gamma \left( \frac{2m}{M} W + \frac{2m}{M} P \frac{\partial W}{\partial P} + \frac{1}{2} \left( \frac{b}{a} \right)^2 \langle p^2 \rangle \frac{\partial^2 W}{\partial P^2} \right), \quad (3.12)$$

where we have dropped terms of order  $m^2/M^2$  and terms involving higher derivatives of  $W$ . Now we note that it is appropriate to identify the dissipation  $\gamma$  as

$$\gamma = \frac{m}{M} \Gamma, \quad (3.13)$$

hence we make a connection between the collision rate and the dissipation. Since  $b/a \approx 2$  to leading order and  $\langle p^2 \rangle = 2mkT$ , we finally obtain the result

$$\frac{\partial W}{\partial t} = -\frac{P}{M} \frac{\partial W}{\partial X} + 2\gamma \frac{\partial(PW)}{\partial P} + 2M\gamma kT \frac{\partial^2 W}{\partial P^2}. \quad (3.14)$$

This is the expected Wigner equation for a system undergoing quantum Brownian motion including dissipation. As is well known, this equation describes the approach to thermal equilibrium. Inverting the Wigner transform, the corresponding density matrix equation is

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2M} \left( \frac{\partial^2 \rho}{\partial x^2} - \frac{\partial^2 \rho}{\partial y^2} \right) - i\hbar\gamma(x - y) \left( \frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial y} \right) - \frac{2M\gamma kT}{\hbar^2} (x - y)^2 \rho. \quad (3.15)$$

Although often given as the master equation for quantum Brownian motion with dissipation [11], this equation is not in fact of the Lindblad form equation (1.2) and actually suffers from a possible small positivity violation [36, 37]. However, it can easily be modified into the Lindblad form equation (1.2) with Lindblad operator equation (1.3) by addition of a term with coefficient proportional to  $1/T$ , so the difference is negligible for high temperatures.

In summary, we obtain the master equation (3.9) with dissipation using momentum and energy conservation to describe the collision, and taking leading and first-order terms in order



$m/M$  together with a simple approximation to incorporate the locality of the interaction. The familiar form equations (3.14), (3.15) are obtained using the further assumption that the Wigner function is a slowly varying function of  $P$  (which, in the density matrix, corresponds to exploring the region of small  $|x - y|$ ).

#### 4. Many-body field theory

We now turn to the second derivation of the master equation, using many-body field theory. This section is based on [14]. We begin by briefly reviewing the formalism [38, 39]. We consider a set of non-relativistic system particles described by a field  $\psi(\mathbf{x})$  interacting through a potential  $\phi(\mathbf{x})$  with an environment described by a field  $\chi(\mathbf{x})$ . The total system is described by the Hamiltonian

$$H = \int d^3x \left( \frac{1}{2M} \nabla \psi^\dagger(\mathbf{x}) \cdot \nabla \psi(\mathbf{x}) + \frac{1}{2m} \nabla \chi^\dagger(\mathbf{x}) \cdot \nabla \chi(\mathbf{x}) \right) + \frac{1}{2} \int d^3x d^3x' \psi^\dagger(\mathbf{x}) \psi(\mathbf{x}') \phi(\mathbf{x} - \mathbf{x}') \chi^\dagger(\mathbf{x}') \chi(\mathbf{x}). \quad (4.1)$$

(For simplicity we set  $\hbar = 1$  hereafter.) In this language, the number densities  $N(\mathbf{x})$  and  $n(\mathbf{x})$  of the system and environment fields are

$$N(\mathbf{x}) = \psi^\dagger(\mathbf{x}) \psi(\mathbf{x}) \quad (4.2)$$

$$n(\mathbf{x}) = \chi^\dagger(\mathbf{x}) \chi(\mathbf{x}). \quad (4.3)$$

The above relations are also more conveniently written in terms of  $a_{\mathbf{k}}$  and  $b_{\mathbf{k}}$ , the annihilation operators for the system and environment, respectively, and the Hamiltonian then is

$$H = \sum_{\mathbf{q}} (E_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + \omega_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}}) + \frac{1}{2V} \sum_{\mathbf{k}' + \mathbf{k}_2 = \mathbf{k}_1 + \mathbf{k}_2} v(\mathbf{k}' - \mathbf{k}_2) a_{\mathbf{k}_1}^\dagger b_{\mathbf{k}_2}^\dagger a_{\mathbf{k}'} b_{\mathbf{k}_2}, \quad (4.4)$$

where  $E_{\mathbf{q}} = \mathbf{q}^2/2M$ ,  $\omega_{\mathbf{q}} = \mathbf{q}^2/2m$ ,  $V$  is the spatial volume of the system (which we assume is in a box) and

$$v(\mathbf{k}) = \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} \phi(\mathbf{x}). \quad (4.5)$$

The Fourier transformed number densities are

$$N_{\mathbf{k}} = \sum_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}+\mathbf{k}} \quad (4.6)$$

$$n_{\mathbf{k}} = \sum_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}+\mathbf{k}}, \quad (4.7)$$

and one may see that the Hamiltonian has the more concise form

$$H = \sum_{\mathbf{q}} (E_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + \omega_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}}) + \frac{1}{2V} \sum_{\mathbf{k}} v(\mathbf{k}) N_{\mathbf{k}} n_{-\mathbf{k}} \quad (4.8)$$

$$= H_0 + H_{\text{int}}. \quad (4.9)$$

From these relations, we see that the environment couples to the number density of the system. It is this feature of many-body field theory that makes it the appropriate medium for the derivation of the master equation emphasizing the role of number density.

The  $S$ -matrix is

$$S = T \exp \left( -i \int_{-\infty}^{\infty} dt H_{\text{int}}(t) \right), \quad (4.10)$$

where

$$H_{\text{int}}(t) = \frac{1}{2V} \sum_{\mathbf{k}} v(\mathbf{k}) N_{\mathbf{k}}(t) n_{-\mathbf{k}}(t) \quad (4.11)$$

and here

$$N_{\mathbf{k}}(t) = \sum_{\mathbf{q}} a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}+\mathbf{k}} e^{i(E_{\mathbf{q}} - E_{\mathbf{q}+\mathbf{k}})t} \quad (4.12)$$

$$n_{\mathbf{k}}(t) = \sum_{\mathbf{q}} b_{\mathbf{q}}^{\dagger} b_{\mathbf{q}+\mathbf{k}} e^{i(\omega_{\mathbf{q}} - \omega_{\mathbf{q}+\mathbf{k}})t}. \quad (4.13)$$

It is enlightening to look at a simple scattering situation to determine how the environment stores information about the system (which in turn determines the preferred basis). Suppose, for simplicity, that the distinguished system is classical and consider what happens when the environment scatters off it. Suppose the environment starts in an initial momentum state  $|\mathbf{k}_0\rangle$  and scatters into a final state  $|\mathbf{k}_f\rangle$ . The scattering amplitude for this process, to first order, is

$$\begin{aligned} \langle \mathbf{k}_f | S | \mathbf{k}_0 \rangle &= \frac{i}{2V} \int_{-\infty}^{\infty} dt \sum_{\mathbf{k}} v(\mathbf{k}) N_{\mathbf{k}}(t) \langle \mathbf{k}_f | n_{-\mathbf{k}}(t) | \mathbf{k}_0 \rangle \\ &= \frac{i}{2V} v(\mathbf{k}) \int dt N_{\mathbf{k}}(t) e^{i(\omega_{\mathbf{k}_f} - \omega_{\mathbf{k}_0})t}, \end{aligned} \quad (4.14)$$

where  $\mathbf{k} = \mathbf{k}_f - \mathbf{k}_0$ . This simple result shows that a single scattering event by the environment stores information about the Fourier transform (in space and time) of the number density. It is in this sense that the number density has a preferred status—this is the variable that is measured most directly by the environment. (An analogous result holds in linear oscillator models [40].)

The measured variables above are of course non-local in time, involving a temporal Fourier transform of the number density, so cannot, in fact, be compatible with a Markovian master equation of the Lindblad form. Under a reasonable slow motion assumption, the system timescale is much slower than the environment timescale, and we may ignore the time dependence in  $N_{\mathbf{k}}(t)$ , yielding

$$\langle \mathbf{k}_f | S | \mathbf{k}_0 \rangle = \frac{i}{2V} v(\mathbf{k}) N_{\mathbf{k}} \delta(\omega_{\mathbf{k}_f} - \omega_{\mathbf{k}_0}). \quad (4.15)$$

This corresponds more directly to a Markovian master equation, as we shall see.

## 5. Derivation of the master equation

Following the method first used by Joos and Zeh [4], we may derive the master equation for the reduced density operator  $\rho$  of the system by considering the scattering of the environment off the system, to second order in interactions. We assume that the system and environment are initially uncorrelated, so the total density operator is

$$\rho_T = \rho_0 \otimes \rho_{\mathcal{E}}. \quad (5.1)$$

We also assume that each scattering event takes place on a timescale which is extremely short compared to the timescale of system dynamics. This means that in an interval of time  $\Delta t$  which is long for the environment but short for the system, we may write

$$\rho_T(t + \Delta t) = S \rho_T(t) S^{\dagger}, \quad (5.2)$$

(where we are using the interaction picture). Expanding (5.2) to second order, the  $S$ -matrix may be written as

$$S = 1 + iU_1 - U_2, \quad (5.3)$$

where

$$U_1 = - \int_{-\infty}^{\infty} dt H_{\text{int}}(t) \quad (5.4)$$

and

$$U_2 = \frac{1}{2} \int dt_1 \int dt_2 T(H_{\text{int}}(t_1)H_{\text{int}}(t_2)). \quad (5.5)$$

The requirement of unitarity,  $S^{-1} = S^\dagger$ , implies that  $U_1 = U_1^\dagger$  and

$$U_2 + U_2^\dagger = U_1^2. \quad (5.6)$$

We will therefore write

$$U_2 = \frac{1}{2}U_1^2 + iB, \quad (5.7)$$

where  $B = B^\dagger$ , so we now have

$$S = 1 + i(U_1 - B) - \frac{1}{2}U_1^2. \quad (5.8)$$

Inserting this in (5.2), we obtain

$$\frac{d\rho_T}{dt} \Delta t = i[U_1 - B, \rho_T] + U_1 \rho_T U_1 - \frac{1}{2}U_1^2 \rho_T - \frac{1}{2}\rho_T U_1^2. \quad (5.9)$$

We now trace equation (5.9) over the environment to obtain the master equation for the system density operator  $\rho$ . As is usual in this sort of model, we assume that the environment is so large that its state is essentially unaffected by the interaction with the system. Since the total density operator starts out in the factored state (5.1), this then means that, to a good approximation,  $\rho_T$  persists in the approximately factored form  $\rho \otimes \rho_\mathcal{E}$ , and we may insert this on the right-hand side of equation (5.9) [41]. We thus obtain the preliminary form for the master equation

$$\frac{d\rho}{dt} \Delta t = i[\text{Tr}_\mathcal{E}(U_1 \rho_\mathcal{E}) - \text{Tr}_\mathcal{E}(B \rho_\mathcal{E}), \rho] + \text{Tr}_\mathcal{E} \left( U_1 \rho_T U_1 - \frac{1}{2}U_1^2 \rho_T - \frac{1}{2}\rho_T U_1^2 \right). \quad (5.10)$$

We now work out these terms in more detail. We will employ the simple but useful slow motion approximation, in which we ignore the time dependence of  $N_{\mathbf{k}}(t)$ . (Corrections to this approximation are considered in [14].) This implies that

$$U_1 \approx -\frac{1}{2V} \sum_{\mathbf{k}} v(\mathbf{k}) N_{\mathbf{k}} \sum_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}-\mathbf{k}} 2\pi \delta(\omega_{\mathbf{q}} - \omega_{\mathbf{q}-\mathbf{k}}). \quad (5.11)$$

The important terms for decoherence are the final three terms on the right-hand side of (5.11). When traced, these give

$$\text{Tr}_\mathcal{E} \left( U_1 \rho_T U_1 - \frac{1}{2}U_1^2 \rho_T - \frac{1}{2}\rho_T U_1^2 \right) = \sum_{\mathbf{k}\mathbf{k}'} c(\mathbf{k}, \mathbf{k}') \left( N_{\mathbf{k}'} \rho N_{\mathbf{k}} - \frac{1}{2}N_{\mathbf{k}} N_{\mathbf{k}'} \rho - \frac{1}{2}\rho N_{\mathbf{k}} N_{\mathbf{k}'} \right), \quad (5.12)$$

where

$$c(\mathbf{k}, \mathbf{k}') = v(\mathbf{k})v(\mathbf{k}') \sum_{\mathbf{q}\mathbf{q}'} \delta(\omega_{\mathbf{q}} - \omega_{\mathbf{q}-\mathbf{k}})\delta(\omega_{\mathbf{q}'} - \omega_{\mathbf{q}'-\mathbf{k}'}) \langle b_{\mathbf{q}}^\dagger b_{\mathbf{q}-\mathbf{k}} b_{\mathbf{q}'}^\dagger b_{\mathbf{q}'-\mathbf{k}'} \rangle_\mathcal{E}. \quad (5.13)$$

We will take the environment to be a thermal state, which is diagonal in the momentum states. It follows that

$$\langle b_{\mathbf{q}}^\dagger b_{\mathbf{q}-\mathbf{k}} b_{\mathbf{q}'}^\dagger b_{\mathbf{q}'-\mathbf{k}'} \rangle_{\mathcal{E}} \propto \delta_{\mathbf{q},\mathbf{q}'-\mathbf{k}'} \delta_{\mathbf{q}',\mathbf{q}-\mathbf{k}}. \quad (5.14)$$

This implies  $\mathbf{k} = -\mathbf{k}'$ , and also that the two delta functions are the same in equation (5.13). We then interpret the square of the delta function in the usual way,

$$\begin{aligned} [\delta(\omega_{\mathbf{q}} - \omega_{\mathbf{q}-\mathbf{k}})]^2 &= \delta(0) \delta(\omega_{\mathbf{q}} - \omega_{\mathbf{q}-\mathbf{k}}) \\ &= \frac{\Delta t}{2\pi} \delta(\omega_{\mathbf{q}} - \omega_{\mathbf{q}-\mathbf{k}}). \end{aligned} \quad (5.15)$$

We now have

$$c(\mathbf{k}, \mathbf{k}') = \delta_{\mathbf{k},-\mathbf{k}'} c(\mathbf{k}) \frac{\Delta t}{2\pi}, \quad (5.16)$$

where

$$\begin{aligned} c(\mathbf{k}) &= \frac{1}{2\pi} |\nu(\mathbf{k})|^2 \sum_{\mathbf{q}} \delta(\omega_{\mathbf{q}} - \omega_{\mathbf{q}-\mathbf{k}}) \langle b_{\mathbf{q}}^\dagger b_{\mathbf{q}-\mathbf{k}} b_{\mathbf{q}-\mathbf{k}}^\dagger b_{\mathbf{q}} \rangle_{\mathcal{E}} \\ &= \frac{1}{2\pi} |\nu(\mathbf{k})|^2 \sum_{\mathbf{q}} \delta(\omega_{\mathbf{q}} - \omega_{\mathbf{q}-\mathbf{k}}) \langle b_{\mathbf{q}}^\dagger b_{\mathbf{q}} \rangle_{\mathcal{E}} (\langle b_{\mathbf{q}-\mathbf{k}}^\dagger b_{\mathbf{q}-\mathbf{k}} \rangle_{\mathcal{E}} + 1). \end{aligned} \quad (5.17)$$

The terms involving environment averages have the usual thermal form (for a bosonic environment)

$$\langle b_{\mathbf{q}}^\dagger b_{\mathbf{q}} \rangle_{\mathcal{E}} = \frac{1}{e^{\beta(\omega_{\mathbf{q}} - \mu)} - 1}, \quad (5.18)$$

where  $\beta = 1/kT$ , with  $T$  temperature, and  $\mu$  is the chemical potential.

The form equation (5.16) means that the important terms in the master equation are of the Lindblad form,

$$\text{Tr}_{\mathcal{E}} \left( U_1 \rho_T U_1 - \frac{1}{2} U_1^2 \rho_T - \frac{1}{2} \rho_T U_1^2 \right) = \Delta t \sum_{\mathbf{k}} c(\mathbf{k}) \left( N_{\mathbf{k}} \rho N_{\mathbf{k}}^\dagger - \frac{1}{2} N_{\mathbf{k}}^\dagger N_{\mathbf{k}} \rho - \frac{1}{2} \rho N_{\mathbf{k}}^\dagger N_{\mathbf{k}} \right), \quad (5.19)$$

where we have used the fact that  $N_{\mathbf{k}}^\dagger = N_{-\mathbf{k}}$ . The remaining two terms in equation (5.10) clearly just modify the unitary dynamics of the system. First we have

$$\text{Tr}_{\mathcal{E}}(U_1 \rho_{\mathcal{E}}) = \frac{1}{2V} \sum_{\mathbf{k}} \nu(\mathbf{k}) N_{\mathbf{k}} \sum_{\mathbf{q}} \langle b_{\mathbf{q}}^\dagger b_{\mathbf{q}-\mathbf{k}} \rangle_{\mathcal{E}} 2\pi \delta(\omega_{\mathbf{q}} - \omega_{\mathbf{q}-\mathbf{k}}). \quad (5.20)$$

Clearly, from the term  $\langle b_{\mathbf{q}}^\dagger b_{\mathbf{q}-\mathbf{k}} \rangle_{\mathcal{E}}$ , this expression will be zero unless  $\mathbf{k} = 0$ , and therefore it is proportional to  $N$ , the total particle number operator (although the overall coefficient will need to be regularized). This therefore contributes a term to the master equation of the form  $[N, \rho]$ . We assume that there is a fixed number of system particles so it is reasonable to take this term to be zero.

The other remaining term in equation (5.10) involves the time-ordering terms in  $U_2$  and is a bit more complicated to evaluate. Fortunately, the detailed form of this expression is not needed here, and it can in fact be easily shown that this term has the form

$$\text{Tr}_{\mathcal{E}}(B \rho_{\mathcal{E}}) = \Delta t \sum_{\mathbf{k}} d(\mathbf{k}) N_{\mathbf{k}} N_{\mathbf{k}}^\dagger \quad (5.21)$$

for some coefficient  $d(\mathbf{k})$  which we will not need. Inserting all these results in equation (5.10), the factors of  $\Delta t$  all drop out, and we obtain, in the Schrödinger picture,

$$\frac{d\rho}{dt} = -i \left[ H_0 - \sum_{\mathbf{k}} d(\mathbf{k}) N_{\mathbf{k}} N_{\mathbf{k}}^{\dagger}, \rho \right] + \sum_{\mathbf{k}} c(\mathbf{k}) \left( N_{\mathbf{k}} \rho N_{\mathbf{k}}^{\dagger} - \frac{1}{2} N_{\mathbf{k}}^{\dagger} N_{\mathbf{k}} \rho - \frac{1}{2} \rho N_{\mathbf{k}}^{\dagger} N_{\mathbf{k}} \right). \quad (5.22)$$

As desired, this is the Lindblad form with the Lindblad operators given by

$$L_{\mathbf{k}} = c^{\frac{1}{2}}(\mathbf{k}) N_{\mathbf{k}}. \quad (5.23)$$

We have therefore produced a derivation of the master equation for a scattering environment which shows very clearly the key role of local number density as the preferred basis, as indicated by the simple scattering calculation, equation (4.15).

It is interesting to note that the decoherence effect is second order in interactions, but we were able to anticipate it from the simple first-order calculation, equation (4.15). The reason for this is the relationship equation (5.7), which shows that the important part of the second-order terms is the square of the first-order terms, and this is a consequence of unitarity.

## 6. Comparison with previous works

It is now important to check that the master equation we have derived reproduces known results when we restrict to the one-particle sector for the system, where a number of derivations have been given in a quantum mechanical framework [4–8]. In the one-particle sector, we may work with a density matrix  $\rho(\mathbf{k}, \mathbf{k}') = \langle \mathbf{k} | \rho | \mathbf{k}' \rangle$ , or equivalently  $\rho(\mathbf{x}, \mathbf{y})$  in the position representation. We use the relations

$$[N_{\mathbf{q}}, a_{\mathbf{k}}^{\dagger}] = a_{\mathbf{k}-\mathbf{q}}^{\dagger} \quad (6.1)$$

$$[N_{\mathbf{q}}, a_{\mathbf{k}}] = -a_{\mathbf{k}+\mathbf{q}}. \quad (6.2)$$

These relations imply that

$$N_{\mathbf{q}} \rho(\mathbf{k}, \mathbf{k}') N_{-\mathbf{q}} = \rho(\mathbf{k} - \mathbf{q}, \mathbf{k}' - \mathbf{q}) \quad (6.3)$$

$$N_{-\mathbf{q}} N_{\mathbf{q}} \rho(\mathbf{k}, \mathbf{k}') = \rho(\mathbf{k}, \mathbf{k}') \quad (6.4)$$

$$\rho(\mathbf{k}, \mathbf{k}') N_{-\mathbf{q}} N_{\mathbf{q}} = \rho(\mathbf{k}, \mathbf{k}'). \quad (6.5)$$

In the position representation, this means

$$N_{\mathbf{k}} \rho(\mathbf{x}, \mathbf{y}) N_{-\mathbf{k}} = e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \rho(\mathbf{x}, \mathbf{y}). \quad (6.6)$$

The master equation for the one-particle density operator  $\rho(\mathbf{x}, \mathbf{y})$  is then

$$\frac{\partial \rho(\mathbf{x}, \mathbf{y})}{\partial t} = -i \langle \mathbf{x} | [H_0, \rho] | \mathbf{y} \rangle - F(\mathbf{x} - \mathbf{y}) \rho(\mathbf{x}, \mathbf{y}), \quad (6.7)$$

where

$$F(\mathbf{x} - \mathbf{y}) = \frac{1}{(2\pi)^5} \int d^3 q d^3 k |v(\mathbf{k})|^2 n_{\mathbf{q}} (n_{\mathbf{q}-\mathbf{k}} + 1) \delta(\omega_{\mathbf{q}} - \omega_{\mathbf{q}-\mathbf{k}}) (1 - e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})}). \quad (6.8)$$

Note that the term involving the coefficient  $d(\mathbf{k})$  in equation (5.22) drops out because  $[N_{\mathbf{k}} N_{\mathbf{k}}^{\dagger}, \rho] = 0$  in the one-particle sector.

Equation (6.7) is of the same general form as earlier results [4–6]. To compare in detail, we first introduce the quantity

$$f(\mathbf{k}, \mathbf{k}') = \frac{m}{2\pi} v(\mathbf{k} - \mathbf{k}'), \quad (6.9)$$

(which appears in the usual Born approximation to first-order scattering). Then, letting  $\mathbf{k} \rightarrow -\mathbf{k} + \mathbf{q}$  in (6.8), we get

$$F(\mathbf{r}) = \frac{1}{(2\pi)^3 m^2} \int d^3 q d^3 k |f(\mathbf{q}, \mathbf{k})|^2 n_{\mathbf{q}}(n_{\mathbf{k}} + 1) \delta(\omega_{\mathbf{q}} - \omega_{\mathbf{k}}) (1 - e^{i(\mathbf{q}-\mathbf{k})\cdot\mathbf{r}}). \quad (6.10)$$

The delta function implies that  $\mathbf{q}^2 = \mathbf{k}^2$ , and we find that

$$F(\mathbf{r}) = \frac{1}{(2\pi)^3 m^2} \int dq q^3 n_q (n_q + 1) \int d\Omega d\Omega' |f(\mathbf{q}, \mathbf{k})|^2 (1 - e^{i(\mathbf{q}-\mathbf{k})\cdot\mathbf{r}}). \quad (6.11)$$

For  $n_q \ll 1$ , and identifying  $(1/2\pi^2)q^2 n_q dq$  as the fraction of particles with momentum magnitude between  $q$  and  $q + dq$ , we find agreement with the careful derivation of Hornberger and Sipe [8], which in turn agrees with recent experiments which measure the decoherence rate [24]. (Hornberger and Sipe corrected erroneous numerical factors in some of the earlier derivations [4, 5], but the qualitative and order of magnitude predictions of these earlier works are correct. See also the cross-check of Adler [26].)

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